# Suggested Solution of Revision Exercise 2

**Question 1.** Show by definition that

(a) $\lim_{x\to 3} \frac{x^3-9}{2x^2-9} = 2$,

(b) $\lim_{x\to 1-} \frac{x}{1-x} = \infty$.

**Solution.** .

(a) We need to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left|\frac{x^3-9}{2x^2-9} - 2\right| < \varepsilon, \quad \text{whenever } 0 < |x-3| < \delta.$$

Note that for any $x \in \mathbb{R}$,

$$\left|\frac{x^3-9}{2x^2-9} - 2\right| = \left|\frac{x^3-4x^2+9}{2x^2-9}\right| = \frac{|x^2-x-3|}{|2x^2-9|} \cdot |x-3|.$$

If $|x-3| < 0.5$, we have $2.5 < x < 3.5$ and $6.25 < x^2 < 12.25$. In this case,

$$-0.25 < x^2 - x - 3 < 6.75 \quad \text{and} \quad 3.5 < 2x^2 - 9 < 15.5.$$

It follows that

$$\left|\frac{x^3-9}{2x^2-9} - 2\right| < \frac{6.75}{3.5} \cdot |x-3| = \frac{135}{7}|x-3|.$$

Let $\varepsilon > 0$. Take $\delta = \min\{0.5, 7\varepsilon/135\}$. Then whenever $|x-3| < \delta$, we have

$$\left|\frac{x^3-9}{2x^2-9} - 2\right| < \frac{135}{7}|x-3| < \frac{135}{7}\delta \le \varepsilon.$$

(b) We need to show that for any $\alpha \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\frac{x}{1-x} > \alpha, \quad \text{whenever } 0 < 1-x < \delta.$$

Note that if $0 < 1-x < \delta \le 1$, we have $0 \le 1-\delta < x < 1$. In this case,

$$\frac{x}{1-x} > \frac{1-\delta}{\delta} \ge 0.$$

Let $\alpha \in \mathbb{R}$. If $\alpha < 0$, we can take any $0 < \delta \le 1$. Then whenever $0 < 1-x < \delta$,

$$\frac{x}{1-x} \ge 0 > \alpha.$$

If $\alpha \ge 0$, take $\delta = 1/(1+\alpha)$. Then whenever $0 < 1-x < \delta$,

$$\frac{x}{1-x} > \frac{1-\delta}{\delta} = \frac{1-1/(1+\alpha)}{1/(1+\alpha)} = \alpha.$$

**Remark.** For the case $\alpha \ge 0$, the choice of $\delta$ is obtained by solving $(1-\delta)/\delta = \alpha$.

**Question 2.** Show that the function $f(x) = 1/x$ is uniformly continuous on $[1, \infty)$, but it is not uniformly continuous on $(0, \infty)$.

**Solution.** To show that $f$ is uniformly continuous on $[1, \infty)$, it suffice to show that $f$ **satisfies a Lipschitz condition** on $[1, \infty)$. i.e., there exists $K > 0$ such that

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| \leq K|x - y|, \quad \text{whenever } x, y \in [1, \infty).$$

I can be done by seeing that for any $x, y \in [1, \infty)$,

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{1}{xy} \cdot |x - y| \leq \frac{1}{1 \cdot 1}|x - y| = |x - y|.$$

($K$ is taken to be 1 implicitly.) To show that $f$ is not uniformly continuous on $(0, \infty)$, we need to show that there exists $\varepsilon_0 > 0$ and two sequences $(x_n)$ and $(y_n)$ in $(0, \infty)$ such that

$$\lim_{n\to\infty}(x_n - y_n) = 0 \quad \text{and} \quad |f(x_n) - f(y_n)| = \left|\frac{1}{x_n} - \frac{1}{y_n}\right| \geq \varepsilon_0, \quad \forall n \in \mathbb{N}.$$

Consider the sequences $(x_n)$ and $(y_n)$ defined by $x_n = 1/n$ and $y_n = 1/(n + 1)$. Then

$$\lim_{n\to\infty}(x_n - y_n) = \lim_{n\to\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{n\to\infty}\frac{1}{n} - \lim_{n\to\infty}\frac{1}{n+1} = 0 - 0 = 0.$$

Also, for any $n \in \mathbb{N}$,

$$\left|\frac{1}{x_n} - \frac{1}{y_n}\right| = |n - (n + 1)| = 1.$$

($\varepsilon_0$ is taken to be 1 implicitly.)

**Remark.** The **Nonuniform Continuity Criteria** (c.f. 5.4.3) is applied to show that $f$ is not uniformly continuous on $(0, \infty)$.

**Question 3.** Let $f : [0, \pi/2] \to \mathbb{R}$ be defined by $f(x) = \sup\{x^2, \cos x\}$. Show that $f$ has an absolute minimum. Moreover, show that if $f$ attains its minimum at $x_0$, then $x_0$ is a solution to the equation $\cos x = x^2$.

**Solution.** Notice that both $x^2$ and $\cos x$ is continuous on $[0, \pi/2]$. By **Homework 7**, the function $f$ is also continous on $[0, \pi/2]$. By the **Maximum-Minimum Theorem** $f$ has an absolute minimum. Now suppose $f$ attains its minimum at $x_0$. Notice that $x^2$ and $\cos x$ is strictly increasing and decreasing respectively on $[0, \pi/2]$.

- For $0 < x < x_0$, we have $x^2 < x_0^2 \leq f(x_0) \leq f(x)$. Therefore $f(x) = \cos x$.
- For $x_0 < x < \pi/2$, we have $\cos x < \cos x_0^2 \leq f(x_0) \leq f(x)$. Therefore $f(x) = x^2$.

Since $f$ is continuous at $x_0$,

$$f(x_0) = \lim_{x \to x_0} f(x) = \lim_{x \to x_0-} f(x) = \lim_{x \to x_0-} \cos x = \cos x_0.$$

On the other hand,

$$f(x_0) = \lim_{x \to x_0} f(x) = \lim_{x \to x_0+} f(x) = \lim_{x \to x_0+} x^2 = x_0^2.$$

It follows that $\cos x_0 = x_0^2$, as they both equal to $f(x_0)$.

**Remark.** The second part of this question relies on the fact that the function $x^2$ is **strictly increasing** on $[0, \pi/2]$ and the function $\cos x$ is **strictly decreasing** on $[0, \pi/2]$. Their proofs are omitted because $x^2$ and $\cos x$ are elementary functions, which are well-studied before. If these facts are not that trivial, we still need to give proofs.

**Question 4.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function with period $p > 0$. i.e., $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Show that

(a) $f$ has an absolute maximum.

(b) $f$ is uniformly continuous on $\mathbb{R}$.

**Solution.** Observe that for any $x \in \mathbb{R}$, there exists (unique) $n \in \mathbb{Z}$ such that $x + np \in [0, p)$.

(a) Consider the continuous function $f$ on $[0, p]$. By **Maximum-Minimum Theorem**, $f$ has an absolute maximum. i.e. there exists $x^* \in [0, p]$ such that

$$f(x) \leq f(x^*), \quad \forall x \in [0, p].$$

It suffices to claim that the above inequality also holds for all $x \in \mathbb{R}$. By the above **observation**, let $n \in \mathbb{Z}$ be such that $x + np \in [0, p)$. Then by the above inequality,

$$f(x) = f(x + np) \leq f(x^*).$$

(b) Consider the continuous function $f$ on $[0, 2p]$. By **Uniform Continuity Theorem**, $f$ is uniformly continuous on $[0, 2p]$. Let $\varepsilon > 0$. There exists $\delta' > 0$ such that

$$|f(x) - f(y)| < \varepsilon, \quad \text{whenever } |x - y| < \delta' \text{ and } x, y \in [0, 2p]. \tag{1}$$

Take $\delta = \min\{\delta', p\}$ and suppose $x, y \in \mathbb{R}$ and $|x - y| < \delta$. By the above **observation**, let $n, m \in \mathbb{Z}$ be such that $x + np, y + mp \in [0, p)$. Notice that

$$|n - m| = \frac{|(x + np) - (y + mp) - (x - y)|}{p} \leq \frac{|(x + np) - (y + mp)| + |x - y|}{p}.$$

Since $x + np, y + mp \in [0, p)$, $|(x + np) - (y + mp)| < p$. Also, $|x - y| < \delta \leq p$. Then

$$|n - m| < \frac{p + p}{p} = 2 \implies m = n - 1, n \text{ or } n + 1.$$

- If $m = n - 1$, then $x + np, y + (m + 1)p \in [0, 2p]$ and

  $$|(x + np) - (y + (m + 1)p)| = |x - y| < \delta \leq \delta'.$$

  Hence by (1), $|f(x) - f(y)| = |f(x + np) - f(y + (m + 1)p)| < \varepsilon$.
- If $m = n$, then $x + np, y + mp \in [0, 2p]$ and

  $$|(x + np) - (y + mp)| = |x - y| < \delta \leq \delta'.$$

  Hence by (1), $|f(x) - f(y)| = |f(x + np) - f(y + mp)| < \varepsilon$.
- If $m = n + 1$, then $x + (n + 1)p, y + mp \in [0, 2p]$ and

  $$|(x + (n + 1)p) - (y + mp)| = |x - y| < \delta \leq \delta'.$$

  Hence by (1), $|f(x) - f(y)| = |f(x + (n + 1)p) - f(y + mp)| < \varepsilon$.

In any cases, we have $|f(x) - f(y)| < \varepsilon$. The result follows.

**Remark.** Although the **observation** is clear to be true, its proof required the Well-Ordering Property on $\mathbb{Z}$. (A bounded below subset of $\mathbb{Z}$ has the least element.) Since it is not a main focus of this problem, the proof is omitted. In part (b), notice that $|x - y| < \delta$ does not implies that $|(x + np) - (y + mp)| < \delta$. Therefore we need to deal with this situation, by using the uniform continuity on $[0, 2p]$ instead of $[0, p]$.

**Question 5.** Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $\mathbb{R}$ with $f(0) = 0$. Prove that there exists some $C > 0$ such that

$$|f(x)| \le 1 + C|x|, \quad \forall x \in \mathbb{R}.$$

(Hint: You may apply the **Well-Ordering Property of** $\mathbb{N}$.)

**Solution.** Since $f$ is uniformly continuous on $\mathbb{R}$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < 1, \quad \text{whenever } |x - y| < \delta.$$

Let $x \in \mathbb{R}$. Consider the set

$$S_x = \Big\{ n \in \mathbb{N} : |x| < n\delta \Big\}.$$

By **Archimedean Property**, there exists $N \in \mathbb{N}$ such that

$$\frac{|x|}{\delta} < N \implies |x| < N\delta.$$

Therefore $S_x$ is a non-empty subset of $\mathbb{N}$. By the **Well-Ordering Property** of $\mathbb{N}$, $S_x$ has a least element $n_0$. i.e., $(n_0 - 1)\delta \le |x| < n_0\delta$. For each $i = 0, 1, 2, ..., n_0$, define

$$x_i = \frac{i}{n_0}x.$$

Then for $i = 1, 2, ..., n_0$,

$$|x_i - x_{i-1}| = \left| \frac{i}{n_0}x - \frac{i-1}{n_0}x \right| = \frac{1}{n_0}|x| < \delta.$$

It follows from the triangle inequality and the definition of $n_0$ that

$$\begin{aligned}
|f(x)| &= |f(x) - f(0)| \\
&= |f(x_{n_0}) - f(x_0)| \\
&\le |f(x_{n_0}) - f(x_{n_0-1})| + |f(x_{n_0-1}) - f(x_{n_0-2})| + \cdots + |f(x_1) - f(x_0)| \\
&= \sum_{i=1}^{n_0} |f(x_i) - f(x_{i-1})| \\
&< \sum_{i=1}^{n_0} 1 \\
&= n_0 \\
&\le 1 + \frac{1}{\delta}|x|
\end{aligned}$$

($C$ is taken to be $1/\delta$ implicitly.)

**Remark.** The idea is to connect 0 and $x$ by points $x_i$'s with distance less than $\delta$.